

# Dyson-Schwinger equations with a parameterized metric

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We construct and solve the Dyson-Schwinger equation (DSE) of quark propagator with a parameterized metric, which connects the Euclidean metric with the Minkowskian one. We show, in some models, the Minkowskian vacuum is different from the Euclidean vacuum. The usual analytic continuation of Green function does not make sense in these cases. While with the algorithm we proposed and the quark-gluon vertex ansatz which preserves the Ward-Takahashi identity, the vacuum keeps being unchanged in the evolution of the metric. In this case, analytic continuation becomes meaningful and can be fully carried out.

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## I. INTRODUCTION

The true metric of space-time is certainly the Minkowskian other than the Euclidean. However, with the Minkowskian metric, in many circumstances, it is very difficult to perform calculations of physical quantities of our interests. Such difficulties are mainly caused by the kinematics which are only included in the Minkowskian system but not in the Euclidean one. Nevertheless, in the framework of path integrals in quantum field theory, the vacuums implied by the Minkowskian metric (with an infinitesimal  $\varepsilon$ -term) and the Euclidean metric should be the same (see, for example, Ref. [1] or section II of this paper). Therefore, if we only focus on some static quantities such as density, condensate and space-like Green function, Euclidean metric and Minkowskian metric are equivalent, giving the same expectation value. However, if we also have interests in the physical quantities which involve in transport processes such as conductance and viscosity, the Minkowskian metric would be inevitable. In recent years, viscosity of strongly coupled quark-gluon plasma (sQGP) [2] and the problem of energy losing during quark jet quenching process in sQGP have attracted a great deal of attention. These problems all involve transport processes and require Minkowskian metric to be implemented [3].

How could we realize Minkowskian metric from an Euclidean system? The most popular way is the analytic continuation (Wick rotation) if we know the concrete analytic form of the Green function. We can easily apply such an approach in perturbative calculations. However, in many non-perturbative cases, for examples, the lattice QCD calculations [4] and the Dyson-Schwinger equation (DSE) approach [5] which are constructed in Euclidean space, we can hardly perform an analytic continuation directly to the numerical results calculated in

Euclidean space. Of course we might use these results to fit an analytic form in some speculations [6, 7]. For instance, a useful priori decomposition rules is the *Källén-Lehmann* spectral representation. However, the problem arises from the lack of a systematic step-by-step program to efficiently assess and improve the accuracy of these speculations.

In many models built in Euclidean space, another even more serious problem bothering us is whether the analytic continuation of Euclidean Green function is meaningful and self-consistent with the principle of quantum field theory, namely whether the Minkowskian path integral formalism and the Euclidean one lead to the same vacuum.

In this paper, we will develop a new scheme, in the framework of DSE, which is free of all the problems mentioned above. The paper is organized as follows. At first, in section II, we introduce the parameterized metric in the context of Quantum Field Theory (QFT) and show how it works as an effective parameter for the task of analytic continuation. In section III, the so-called vacuum problem, which bothers us in some models, is described explicitly. Then, we construct Dyson-Schwinger equation for quark with the parameterized metric in section IV. The numerical algorithm and calculated results are given in section V. Finally, in section VI, we make a summary and give some remarks.

## II. PARAMETERIZED METRIC AND ITS VALIDITY IN SOME SIMPLE CASES

To introduce a parameterized metric, we should deal with the metric as a variable everywhere in QFT. For example, we consider the simple  $\phi^4$  theory. Its action can be expressed as

$$S = \int d^4x \sqrt{-\det g_{\mu\nu}} \left\{ -\frac{1}{2} g^{\mu\nu} \frac{\partial \phi}{\partial x^\mu} \frac{\partial \phi}{\partial x^\nu} - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4!} \phi^4 \right\}. \quad (1)$$

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The covariant Minkowskian/Euclidean metric adopted in this paper is  $g_{\mu\nu}^M = \text{diag}(-1, +1, +1, +1)$  and  $g_{\mu\nu}^E = \text{diag}(+1, +1, +1, +1)$  and the contravariant metric satisfies  $g^{\mu\lambda}g_{\lambda\nu} = \delta_\nu^\mu$ . Considering the requirements of  $\sqrt{-\det g_{\mu\nu}^E} = -i$  and  $\sqrt{-\det g_{\mu\nu}^M} = 1$ , we could parameterize the covariant metric as

$$g_{\mu\nu}(\varepsilon) = \begin{pmatrix} e^{i\varepsilon} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (2)$$

with  $\varepsilon \in [0, \pi)$ . Then we have  $g(\varepsilon = 0) \equiv g^E$  and  $g(\varepsilon \rightarrow \pi) \rightarrow g^M$ . Furthermore, one could easily prove that  $iS_M[\phi] = iS[g(\pi), \phi]$  and  $-S_E[\phi] = iS[g(0), \phi]$ , so that the uniform path-integral formalism reads

$$\int [d\phi] e^{iS[g(\varepsilon), \phi]}. \quad (3)$$

Up to now, we have only found a simple way (namely, varying the parameter  $\varepsilon$ ) to connect  $g_M$  with  $g_E$ . In fact, there are infinite ways to realize this connection. Why ours is preferred, or what does it mean? To answer these questions, as an simple example, we construct the quantum theory of the  $\phi^4$  model from the Lagrangian

$$L = \sqrt{-\det g_{\mu\nu}(\varepsilon)} \left\{ -\frac{1}{2} g^{\mu\nu}(\varepsilon) \frac{\partial \phi}{\partial x^\mu} \frac{\partial \phi}{\partial x^\nu} - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4!} \phi^4 \right\}.$$

We will use the Hamiltonian formulation to quantize it. The momentum density conjugate to  $\phi(x)$  is

$$\pi(x) \equiv \frac{\partial L}{\partial \dot{\phi}} = -g^{00}(\varepsilon) \sqrt{-\det g_{\mu\nu}(\varepsilon)} (\dot{\phi}). \quad (4)$$

By using the relation  $\det g_{\mu\nu} = g_{00} = \frac{1}{g^{00}}$ , we obtain the Hamiltonian

$$H \equiv \int d^3x [\pi \dot{\phi} - L] = e^{i(\varepsilon - \pi)/2} \int d^3x \left[ \frac{1}{2} \pi^2 + \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4!} \phi^4 \right]. \quad (5)$$

At  $t = 0$ , we suppose that  $\phi(\vec{x})$  and  $\pi(\vec{x})$  are Hermite operators and satisfy the relation

$$\begin{aligned} [\phi(\vec{x}), \pi(\vec{y})] &= i\delta^{(3)}(\vec{x} - \vec{y}), \\ [\phi(\vec{x}), \phi(\vec{y})] &= [\pi(\vec{x}), \pi(\vec{y})] = 0, \end{aligned} \quad (6)$$

and the equation of motion for Heisenberg operator is

$$O(t) = U(t)^{-1} O(0) U(t), \quad (7)$$

where the evolution matrix is

$$U(t) = e^{-iHt} = e^{-iH_M \hat{t}}. \quad (8)$$

Here,  $H_M$  is the Hamiltonian operator in Minkowskian space as the same as that commonly used, and  $\hat{t} \equiv e^{i(\varepsilon - \pi)/2} t$ . We should note that  $U(t)$  is not a unitary matrix since the  $H$  is not Hermite. So the operators

$\phi(t)$  and  $\pi(t)$  need not to be Hermite when  $t \neq 0$ , but it does not affect Eq. (6) to be held at  $t \neq 0$ . In fact, if  $\phi(t = 0)$  is Hermite, from Eqs. (4), (7) and (8) we can prove that  $\pi(t = 0)$  is also Hermite (note that  $\dot{\phi}(0)$  is not Hermite). Moreover, one could easily prove that the correlation function  $G(x, y)$  in vacuum is time( $t$ )-translation invariant. From these observations, we conclude that our construction is self-consistent.

In addition, there are some other properties that we need for our purpose in this paper. Firstly, the structure of eigenstates of the  $H$  is independent of  $\varepsilon$ . Secondly, the effects of the parameterized metric we adopt is equivalent to the Wick rotation to time axis in complex plane but nothing else. In fact, one prefers to regard Eq. (8) as the standard definition of Wick rotation and take it to realize analytic continuation of Green functions. However, in path-integral formalism, our method is more convenient. As we know that the path-integral formalism can automatically perform the time order operator, but the in/out states are unfixed in it. Nevertheless, the parameterized metric undertakes an extra responsibility, besides the Wick rotation, that guarantees the in/out states to be those in vacuum (the lowest energy state of  $H_M$ ). It is true in our method with varying  $\varepsilon \in [0, \pi)$ . To see it more explicitly, we just need to show that

$$U(\infty)|\alpha\rangle \propto e^{-H_M \cos(\varepsilon/2)\infty} |\alpha\rangle \rightarrow |vac\rangle. \quad (9)$$

It should be emphasized that Eq. (9) is definitely true except for the case with  $\varepsilon = \pi$ , so that we could only take the limit  $\varepsilon \rightarrow \pi$  to realize Minkowskian space whose effects are equivalent to the  $\varepsilon$ -term trigger adopted commonly (see, for example, Ref. [1]). Until now, we could say that this parameterized metric, used in path-integral formalism for analytic continuation of Green functions, is legitimate. (In addition, Eq. (9) means that the vacuum state implied by  $g(\varepsilon)$  is identical and independent of  $\varepsilon$ . However, as we will see in the next section, such a statement is not always satisfied by models. We refer it as vacuum problem.)

As a simple test of our method, we implement it to calculate the one-loop electron self-energy. At first, we need conventions to simplify our representation. In this paper, real space-time variables are always thought to be the components in a contravariant vector  $x^\mu$ . Naturally, the real momentum variables are always thought to be those in a covariant vector  $p_\mu$ . The Fourier transform is

$$f(x^\mu) = \int \frac{d^4 p_\mu}{(2\pi)^4} \sqrt{-\det g^{\mu\nu}(\varepsilon)} e^{-ip_\mu x^\mu} f(p_\mu), \quad (10)$$

$$f(p_\mu) = \int d^4 x^\mu \sqrt{-\det g_{\mu\nu}(\varepsilon)} e^{ip_\mu x^\mu} f(x^\mu). \quad (11)$$

Since we are considering Fermion, the Dirac matrix  $\gamma^\mu$  should also be parameterized to satisfy

$$\begin{aligned} \{\gamma^\mu(\varepsilon), \gamma^\nu(\varepsilon)\} &= 2g^{\mu\nu}(\varepsilon), \\ \{\gamma_\mu(\varepsilon), \gamma_\nu(\varepsilon)\} &= 2g_{\mu\nu}(\varepsilon), \\ \{\gamma^\mu(\varepsilon), \gamma_\nu(\varepsilon)\} &= 2g_\nu^\mu(\varepsilon) = 2\delta_\nu^\mu. \end{aligned} \quad (12)$$

With the parameterized metric, the electron self-energy at the one-loop level can be written as

$$\Sigma_2(p) = ie^2 \int \frac{d^4 q}{(2\pi)^4} \sqrt{-\det g^{ab}} \gamma^\mu \frac{-i\not{q} + m_e}{q^2 + m_e^2} \gamma^\nu \frac{g_{\mu\nu}}{k^2 - \mu^2}, \quad (13)$$

where

$$\begin{aligned} \not{q} &= g_{\alpha\beta} \gamma^\alpha q^\beta, \\ q^2 &= g_{\alpha\beta} q^\alpha q^\beta, \\ k^2 &= g_{\alpha\beta} k^\alpha k^\beta. \end{aligned} \quad (14)$$

Using the Pauli-Villars regularization procedure, we have

$$\begin{aligned} \Sigma_2(p) &= ie^2 \int \frac{d^4 q}{(2\pi)^4} \sqrt{-\det g^{ab}} \gamma^\mu \frac{-i\not{q} + m_e}{q^2 + m_e^2} \gamma^\nu \\ &\quad \times g_{\mu\nu} \left( \frac{1}{k^2} - \frac{1}{k^2 + \Lambda^2} \right). \end{aligned} \quad (15)$$

Then

$$S_e^{-1}(p) = i\not{p} + m_e - \Sigma_2(p) \quad (16)$$

$$= i\not{p} A(p^2) + B(p^2), \quad (17)$$

where the functions  $A$  and  $B$  can be written explicitly as

$$\begin{aligned} A(p^2) &= 1 + e^2 \int \frac{d^4 q}{(2\pi)^4} (e^{-i\varepsilon/2}) \frac{1}{q^2 + m_e^2} \frac{2p \cdot q}{p^2} \\ &\quad \times \left( \frac{1}{k^2} - \frac{1}{k^2 + \Lambda^2} \right), \end{aligned} \quad (18)$$

$$\begin{aligned} B(p^2) &= m_e + e^2 \int \frac{d^4 q}{(2\pi)^4} (e^{-i\varepsilon/2}) \frac{4m_e}{q^2 + m_e^2} \\ &\quad \times \left( \frac{1}{k^2} - \frac{1}{k^2 + \Lambda^2} \right). \end{aligned} \quad (19)$$

which can be calculated by proper numerical integral.

Note that  $\Sigma_2(p)$  can also be obtained through standard perturbative procedure, which reads

$$\begin{aligned} \Sigma_2(p) &= -\frac{\alpha}{4\pi} \int_0^1 dx [2i(1-x)\not{p} + 4m_e] \\ &\quad \times \ln \left( \frac{(1-x)\Lambda^2}{x(1-x)p^2 + xm_e^2} \right). \end{aligned} \quad (20)$$

From Eq. (20), we can also extract the structure functions  $A(p^2)$  and  $B(p^2)$ , where  $p^2$  should be understood as complex number but not that in Eq. (14).

To illustrate the validity of the parameterized metric, we take the analytical structure of the mass function  $M(p^2) = B(p^2)/A(p^2)$  as an example. The calculated results in our presently proposed parameterized metric and that in the standard perturbative procedure are displayed in Fig.1. Looking over Fig.1, one can recognize definitely that our presently proposed parameterized metric method is equivalent to the traditional perturbative calculations.

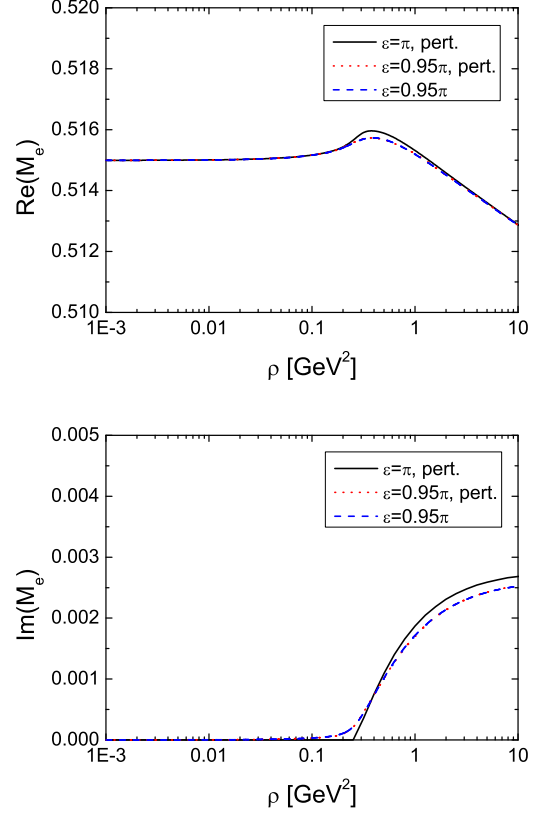


FIG. 1: (color online) Calculated result of the analytical structure of the mass function in our parameterized metric and the comparison with those given in the standard perturbative procedure, where  $p^2 = \rho e^{-i\varepsilon}$ . The solid line is that given by perturbative theory in the exact Minkowskian space (with  $\varepsilon = \pi$  in the expression of  $p^2$ ), the dot line is also that calculated in perturbative theory but with  $\varepsilon = 0.95\pi$ , the dash line is that obtained with our parameterized metric at  $\varepsilon = 0.95\pi$ .

It should be noted that, in this example, the equivalence between our newly proposed method and the standard perturbative calculation is somewhat trivial and our method is not economic. However, in non-perturbative frameworks, traditional analytic continuation does not work anymore (it requires us knowing the strict analytical expression of the solutions), instead, our method will become valuable and efficient. Anyway, such an equivalence indicates that presently proposed parametrized metric method is absolutely valid. Then, as we will see below, we can successfully apply this method to Dyson-Schwinger equation. However, before we do that, there is another problem deserving our attention, namely the vacuum problem, because it will influence the proper truncation of Dyson-Schwinger equation.

### III. VACUUM PROBLEM

To demonstrate the vacuum problem clearly, we first consider a free quark system with finite mass and chemical potential. In this case, the analytical expression for quark propagator is exactly known. The total influence of chemical potential  $\mu$  on the quark propagator is just a complex continuation of  $p_0$  ( $p_0 \rightarrow p_0 - i\mu$ ). Actually, in many articles,  $\mu$  is used as a single parameter for analytic continuation and not referred to as chemical potential. When  $\mu > m$ , Fermi-surface will emerge from the origin of momentum space, and the vacuum will be changed. However, since there is no interaction, the influence of the chemical potential  $\mu > m$  on the vacuum can not alter the quark propagator's form. Now, we turn on some interaction which needs not to be strong (see, for example, Refs. [7–9]). In this case, the quark propagator will include self-energy part which is dependent on the vacuum. Therefore, as the vacuum has been altered, the quark propagator with  $\mu > m$  is no longer an analytically continued Green function with  $p_0 \rightarrow p_0 - i\mu$ . Up to now, we have not questioned the meaning of analytic continuation. We could just say that, beyond the mass threshold, chemical potential is no longer a valid parameter for the task of the analytic continuation [7].

However, in reality, there might be even worse situation when we try to use the parameterized metric  $g_{\mu\nu}(\varepsilon)$  to realize the analytic continuation. As we will see, with the increasing of the parameter  $\varepsilon$ , the range of  $p^2 = p_\mu p_\nu g^{\mu\nu}$  ( $p_\mu \in R$ ) will become larger and larger in the complex  $p^2$  plane. Considering the Minkowskian limit  $\varepsilon \rightarrow \pi$ , the maximal range of the complex  $p^2$  would be in the lower half-plane except the time-like real axis. Once a pole enters the range of  $p^2$ , the situation becomes very similar to what happened in the case of large chemical potential  $\mu > m$ . The vacuum will be changed from then on. Because of the interaction, Green function is no longer an analytic function of  $p^2$ . In this case, how could we determine the physical one naïvely? Can we still say that we have chosen a wrong way of analytic continuation as the same as that for chemical potential? In the sense of analytic continuation, we are indeed wrong. However, at the same time, we also see that the results of analytic continuation (if exists) is no longer corresponding to the true vacuum structure implied by the Minkowskian metric. It will result in a serious contradiction. On one hand, choosing the strict analytic continuation means we have to abandon the Minkowskian metric which is our original and real purpose! On the other hand, choosing a complete Minkowskian framework, we will lose the analyticity of Green function! Obviously, the root of this problem is the presence of singularities outside of the time-like real axis of  $p^2$  plane. Actually, in this case, the Minkowskian vacuum is different from the Euclidean one. For example, if the quark propagator has a pair of complex poles

with  $\text{Im}(z) \neq 0$  (see, for example, Ref. [10]),

$$S(p) = \frac{1}{i\gamma \cdot p + z} + \frac{1}{i\gamma \cdot p + z^*}, \quad (21)$$

the chiral quark condensate reads

$$- \langle \bar{q}q \rangle = 2N_c Z_4 \int \frac{d^3 p}{(2\pi)^3} f_2(|\vec{p}|), \quad (22)$$

with the 3-momentum distribution function

$$f_2(|\vec{p}|) = \frac{1}{4\pi} \int_{-\infty}^{\infty} dp_4 \text{tr}_D S(p). \quad (23)$$

For our purpose, we just need to evaluate the Euclidean  $f_2^E(0)$  and the Minkowskian  $f_2^M(0)$

$$f_2^E(0) = C \int_{-\infty}^{\infty} dp_4 \left[ \frac{z}{p_4^2 + z^2} + \frac{z^*}{p_4^2 + (z^*)^2} \right], \quad (24)$$

$$f_2^M(0) = C \int_{-\infty}^{\infty} idp_4 \left[ \frac{z}{p_4^2 - z^2 + i0^+} + \frac{z^*}{p_4^2 - (z^*)^2 + i0^+} \right]. \quad (25)$$

After some direct calculations, we obtain the difference between the distribution functions of the condensate in the two metrics as

$$f_2^E(0) - f_2^M(0) = 2\pi C \neq 0. \quad (26)$$

Then the chiral condensate takes definitely different values in the two metrics.

Since  $\bar{q}q$  is a gauge invariant local operator, different condensates (or its distribution) must come from different physical vacuums. It is easy for us to accept a different vacuum induced by the chemical potential  $\mu > m$ . However, in the framework of path integrals in quantum field theory, the vacuums implied by the Minkowskian metric ( $\varepsilon \rightarrow \pi$ ) and the Euclidean metric must be the same as we have already seen in section II. So, if a model is well defined, the contradiction mentioned above should not exist. It means that the singularities of the propagator can not exist outside of the time-like real axis of  $p^2$ . With this restriction, the parameterized metric can help us to perform the analytic continuation successfully and will indeed preserve the vacuum until  $g^{\mu\nu}(\varepsilon)$  approaches infinitely the Minkowskian metric.

In the following, we will try to establish such an approach in the framework of the Dyson-Schwinger equations [5].

### IV. QUARK'S DSE MODELS WITH THE PARAMETERIZED METRIC

In this paper, the quantity of our interests is the quark propagator in momentum space

$$S(p) = \int d^4 x \sqrt{-\det g_{\mu\nu}} e^{ip \cdot x} \langle \Omega | T q(x) \bar{q}(0) | \Omega \rangle. \quad (27)$$

It satisfies the quark Dyson-Schwinger equation [5]

$$S^{-1}(p) = (i\gamma \cdot p + m) + \Sigma(p), \quad (28)$$

with

$$\Sigma(p) = \int_q^\Lambda \sqrt{-\det g^{\mu\nu}} g^2 D^{\mu\nu}(k) \frac{\lambda^a}{2} \gamma_\mu S(q) \Gamma_\nu^a(q, p). \quad (29)$$

And according to its Lorentz structure, it is usually decomposed as

$$S^{-1}(p) = i\gamma \cdot p A(p^2) + B(p^2), \quad (30)$$

or

$$S(p) = i\gamma \cdot p \sigma_A(p^2) + \sigma_B(p^2). \quad (31)$$

Since we will make use of a model that is free of ultraviolet divergence, here, we need not to discuss the issues related to the regularization and renormalization of Dyson-Schwinger equation [11, 12]. Anyway, if one adopts a model that has ultraviolet divergence, Pauli-Villars regularization would be a reasonable choice just as we have done in section II.

Because the quark DSE involves effective (full) gluon propagator and quark-gluon interaction vertex which can not be determined by quark DSE itself, we need to choose some models for them. In Landau gauge, we have

$$D^{\mu\nu}(k) = D(k) [g^{\mu\nu}(\varepsilon) - \frac{k^\mu k^\nu}{k^2}], \quad (32)$$

with  $k^\mu \equiv g^{\mu\nu} k_\nu$ . And for the vertex  $\Gamma_\mu^a$ , Abelian assumption gives

$$\Gamma_\mu^a(q, p) = \frac{\lambda^a}{2} \Gamma_\mu(q, p). \quad (33)$$

Finally, we need to choose some proper models for  $D(k)$  and  $\Gamma_\mu(q, p)$ . In this paper, to preserve the vacuum in the evolution of the metric from Euclidean to Minkowskian, we have to adopt a self-consistent gluon propagator whose singularities do not appear outside of the time-like  $k^2 < 0$  axis. As a simple choice, we take

$$g^2 D(k) = 4\pi^2 D \frac{\chi^2}{(k^2 + \Delta)^2}, \quad (34)$$

in this paper. Since gluon is confined in reality, one might have instinctive suspicion on our choice in sense of confinement. Nevertheless, the lack of *Källén-Lehmann* spectral representation is not the only way to exclude a particle from the physical Hilbert space. For example, in QED, the longitudinal photon indeed have legitimate *Källén-Lehmann* spectral representation, but it is still located outside of the physical Hilbert space. Of course, we are far away from the solution of QCD confinement [13–15]. People need to make much more efforts for this problem. More or less, it has gone beyond the

main scope of our current discussions. Anyway, we will come back to this issue later.

For the quark-gluon interaction vertex, the simplest choice is the so called rainbow approximation

$$\Gamma^\mu = \gamma^\mu. \quad (35)$$

In many practical calculations, the naïve approximation has got triumphs. However, given the Abelian assumption, the bare quark-gluon interaction vertex violates the Ward-Takahashi identity (WTI) which is the consequence of the conservation of quark current. During a single-quark process, current's conservation is crucial for the unitary. Therefore, rainbow approximation could not promise that singularities of quark propagator can only appear at the time-like real axis ( $p^2 < 0$ ).

Beyond the bare (or rainbow) approximation, people have developed other vertex forms that satisfy the WTI. It has been shown that the so-called Ball-Chiu (BC) ansatz for the vertex [16] is a successful form of them (see, for example, Ref. [17]). It is expressed as

$$\Gamma_\mu^{BC} = \Sigma_A \gamma_\mu + \frac{\gamma \cdot (p+q)}{2} \Delta_A (p+q)_\mu + i \Delta_B (p+q)_\mu, \quad (36)$$

with

$$\Sigma_A = \frac{A(p^2) + A(q^2)}{2}, \quad (37)$$

$$\Delta_A = \frac{A(p^2) - A(q^2)}{p^2 - q^2}, \quad (38)$$

$$\Delta_B = \frac{B(p^2) - B(q^2)}{p^2 - q^2}. \quad (39)$$

Since BC vertex fully satisfies WTI, we expect that singularities of quark propagator could only appear at the time-like real axis ( $p^2 < 0$ ) (similar conclusion has already been drawn in Ref. [6]). Then, we will take BC vertex in our practical calculation.

## V. NUMERICAL CALCULATION AND RESULTS

### A. Algorithm and its validity

For numerical calculation, we extract at first the equations for the functions  $A$  and  $B$  from the DSE of quark. After some derivation, we can write the equations about the  $A$  and  $B$  in the parametrized metric as

$$A(p^2) = 1 + \int e^{-\frac{i\varepsilon}{2}} dq_0^2 d\mathbf{q}^2 \frac{|\mathbf{q}|}{4|q_0|} \frac{\Theta_A[A(p^2), A(q^2), p^2, q^2]}{q^2 A^2(q^2) + B^2(q^2)}, \quad (40)$$

$$B(p^2) = \int e^{-\frac{i\varepsilon}{2}} dq_0^2 d\mathbf{q}^2 \frac{|\mathbf{q}|}{4|q_0|} \frac{\Theta_B[A(p^2), A(q^2), p^2, q^2]}{q^2 A^2(q^2) + B^2(q^2)}, \quad (41)$$

with

$$\begin{aligned}\Theta_A &= A(q^2)\Sigma_A F_A^1 + A(q^2)\Delta_A F_A^2 + B(q^2)\Delta_B F_A^3, \\ \Theta_B &= B(q^2)\Sigma_A F_B^1 + B(q^2)\Delta_A F_B^2 + A(q^2)\Delta_B F_B^3,\end{aligned}$$

where  $q^2 = q_0^2 e^{-i\varepsilon} + \mathbf{q}^2$ , and  $F_{A,B}^i (i = 1, 2, 3)$  are functions of  $p^2$  and  $q^2$  because the angular integrals have been done. These simplified equations are the starting point of our further numerical calculations.

Note that a given metric  $g^{\mu\nu}(\varepsilon)$  determines a specific region of  $p^2$  on complex plane which can be shown in Fig. 2. Then we can parameterize  $p^2$  as  $\rho e^{-i\theta}$  with  $\rho \in [0, \infty)$  and  $\theta \in [0, \varepsilon]$ . Obviously, when  $\varepsilon$  approaches  $\pi$ , the region of  $p^2$  plus its conjugation will cover the whole complex plane except the time-like axis. It should be mentioned that, if we take the translation of  $p_0 \rightarrow p_0 - i\mu$  to perform the analytic continuation, the range of  $(p_0 - i\mu)^2 + \vec{p}^2$  is bounded by a parabolic curve. Even if singularities only appear at time-like real axis, such parabolic curve will inevitably encounter the singularities as  $\mu$  become moderately large. In this case, the available analytic information is only the inner part bounded by the parabola which can not include any singularity. However, with the present parameterized metric, we can easily overcome such a problem.

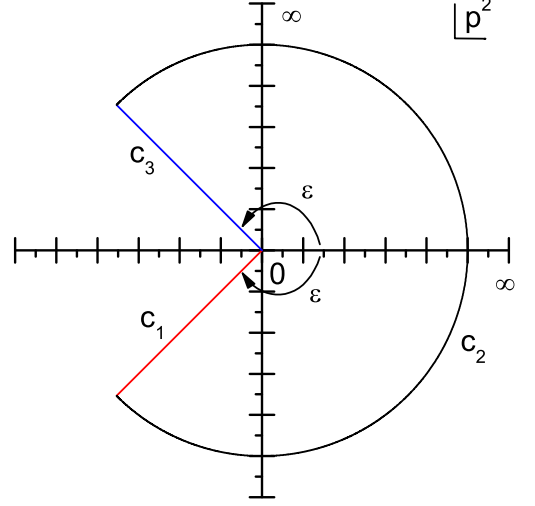


FIG. 2: [color online] Schematics of the area and the contour of the complex momentum  $p^2$  (the lower part) and  $p^{2*}$  (the upper part) with the parameterized metric  $g^{\mu\nu}(\varepsilon)$ .

With above representations, we obtain

$$\begin{aligned}A(\rho e^{-i\theta}) &= 1 + (e^{-i\varepsilon/2}) \int_0^\infty \rho' d\rho' \int_0^\varepsilon d\theta' \left( \frac{1}{4\sin\varepsilon} \sqrt{\frac{\sin(\varepsilon - \theta')}{\sin\theta'}} \right) \frac{\Theta_A[A(\rho e^{-i\theta}), A(\rho' e^{-i\theta'}), \rho e^{-i\theta}, \rho' e^{-i\theta'}]}{(\rho' e^{-i\theta'}) A^2(\rho' e^{-i\theta'}) + B^2(\rho' e^{-i\theta'})}, \\ B(\rho e^{-i\theta}) &= (e^{-i\varepsilon/2}) \int_0^\infty \rho' d\rho' \int_0^\varepsilon d\theta' \left( \frac{1}{4\sin\varepsilon} \sqrt{\frac{\sin(\varepsilon - \theta')}{\sin\theta'}} \right) \frac{\Theta_B[A(\rho e^{-i\theta}), A(\rho' e^{-i\theta'}), \rho e^{-i\theta}, \rho' e^{-i\theta'}]}{(\rho' e^{-i\theta'}) A^2(\rho' e^{-i\theta'}) + B^2(\rho' e^{-i\theta'})}.\end{aligned}\quad (42)$$

In principle, the above equations can be solved directly. In practice, to simplify numerical calculations, we consider ultraviolet boundary conditions for the functions  $A$  and  $B$  and the Cauchy integral formulas [10] which read

$$A(C_3) = A^*(C_1), \quad B(C_3) = B^*(C_1), \quad (43)$$

$$A(C_2) = 1, \quad B(C_2) = 0, \quad (44)$$

$$F(z) = \frac{1}{2\pi i} \oint_{C_1+C_2+C_3} \frac{F(t)}{t-z} dt, \quad (45)$$

where the paths  $C_1$ ,  $C_2$  and  $C_3$  are those shown in Fig. 2, and  $F$  stands for the functions  $A$  and  $B$ . Then, in the integral equations, the only unknown function is  $F(C_1)$ , or  $F(\rho e^{-i\varepsilon})$  which can be fully solved. Therefore, the original two-dimensional problem is reduced to a one-dimensional one. Such a reduction is very significant because it makes the numerical calculation with large  $\varepsilon$  ( $\rightarrow \pi$ ) possible.

To check the validity of our method, we adopt firstly a relatively small  $\varepsilon = 3\pi/4$  to solve the equation (with

$D = 0.25\text{GeV}^2$ ,  $\chi = 1.0\text{GeV}^2$  and  $\Delta = 0.1\text{GeV}^2$ ). The solutions of the functions  $A$  and  $B$  are illustrated in Fig. 3. Making use of these solutions, we can generate the data at  $\theta = 0$  with the Cauchy integral formulas. We compare then the generated data with the solutions obtained by solving directly the equations in Euclidean space (solving the equations after taking  $\varepsilon = 0$ ). The comparison is displayed in Fig. 4. The figure manifests apparently that the two groups of curves are exactly co-incident. It indicates that the validity of our method is in very high precision and the vacuum really remains against the evolution of  $\varepsilon$  from 0 to  $0.75\pi$ .

To clearly illustrate the self-consistency, we have to show that the quark propagator has no pole in the corresponding area. We display then the obtained quark propagator (actually the absolute value of the  $\sigma_B$ , denoted as  $|\sigma_B|$ , is sufficient for this checking) in the whole range of  $p^2$  in Fig. 5. It is apparent that there is indeed no pole located in the range of  $p^2$ .

In principle, for a given  $\varepsilon$ , as long as the Green func-

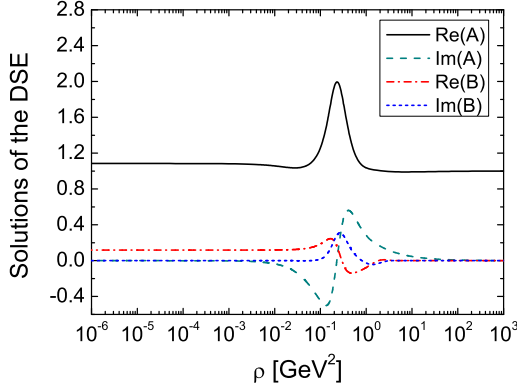


FIG. 3: [color online] Calculated variation behaviors of  $A(\rho e^{-i\varepsilon})$  and  $B(\rho e^{-i\varepsilon})$  with respect to  $\rho$  at  $\varepsilon = 3\pi/4$ .

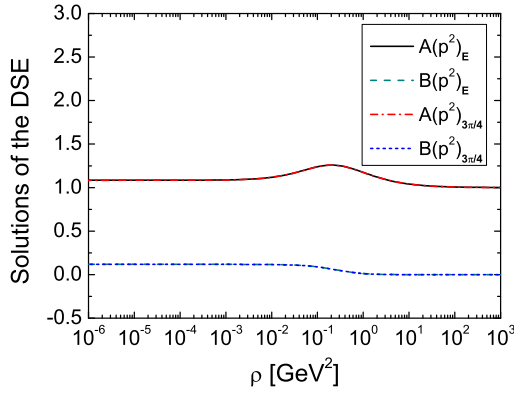


FIG. 4: [color online] The comparison between the data generated by our algorithm and those obtained directly in the Euclidean space.

tions have no singularity in the involved momentum region, our results would be reliable. As we have mentioned above, the singularities of the quark propagator could only appear at the time-like axis, then the analytic continuation on the whole  $p^2$  plane except the time-like axis could be achieved by the limit  $\varepsilon \rightarrow \pi$ .

### B. Almost Minkowskian propagator

In this subsection, we discuss the quark propagator in the case that is very close to the Minkowskian by performing the calculations with increasing the  $\varepsilon$  to a value highly close to  $\pi$ . It should be noted that, the true limit ( $\varepsilon = \pi$ ) is very hard to achieve (and not necessary) in practice. At our current level, the largest  $\varepsilon$  that can yield reliable results is  $0.95\pi$ . In addition, we consider the dependence of the quark propagator on the coupling strength  $D$  and the screening width  $\Delta$ .

After having solved the quark DSE with our algo-

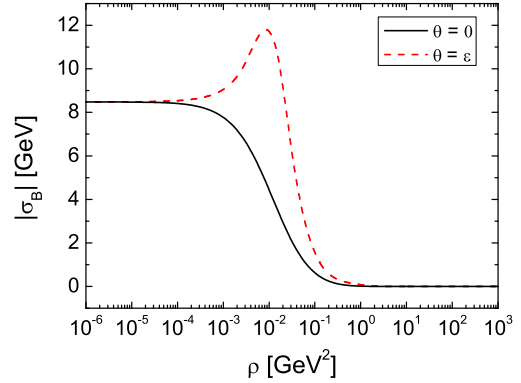
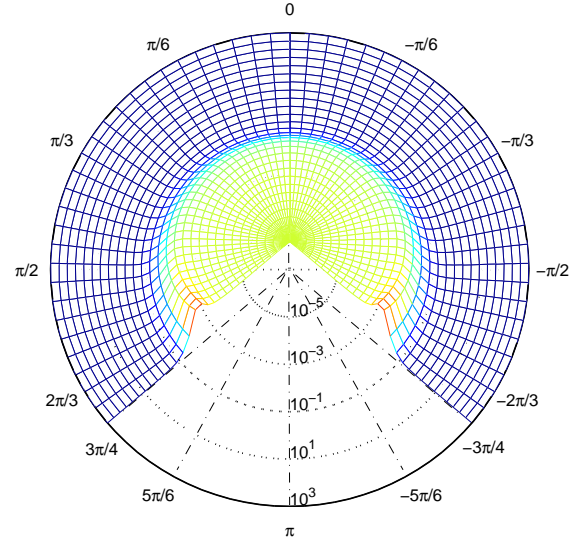


FIG. 5: [color online] Calculated variation behavior of the scalar part of the quark propagator  $|\sigma_B|$  in the whole momentum area in the case of  $\varepsilon = 3\pi/4$  (upper panel) and that at two special values of  $\theta$  (lower panel).

rithm, we first analyze the behaviors of the functions  $A(p^2)$  and  $B(p^2)$  in the quark propagator against the parameter  $\varepsilon$  and with fixed parameter set  $D = 1.06\text{GeV}^2$ ,  $\chi = 1.0\text{GeV}^2$  and  $\Delta = 0.5\text{GeV}^2$ . The obtained results of the real part of the functions  $A$  and  $B$  at several values of  $\varepsilon$  are illustrated in the upper, middle panel of Fig. 6, respectively, and both the real and imaginary parts of the  $A$  and  $B$  at the maximal value of  $\varepsilon$  we reached in the lower panel of Fig. 6. We can notice apparently from the figure that both the functions  $A$  and  $B$  have a peak at some complex momentum. Moreover, the peaks grow sharper and sharper with the increasing of  $\varepsilon$ . Although we have to stop at  $\varepsilon = 0.95\pi$  because of numerical difficulty (the peaks make the numerical contour integral inexact as the  $\varepsilon$  approaches much more to  $\pi$ ), we can predict that the peaks will finally evolve into singularities. Namely, the analytic functions  $A(p^2)$  and  $B(p^2)$  both have poles at  $p_g^2$  (on the time-like momentum axis).

To show the limit behavior of the imaginary part of



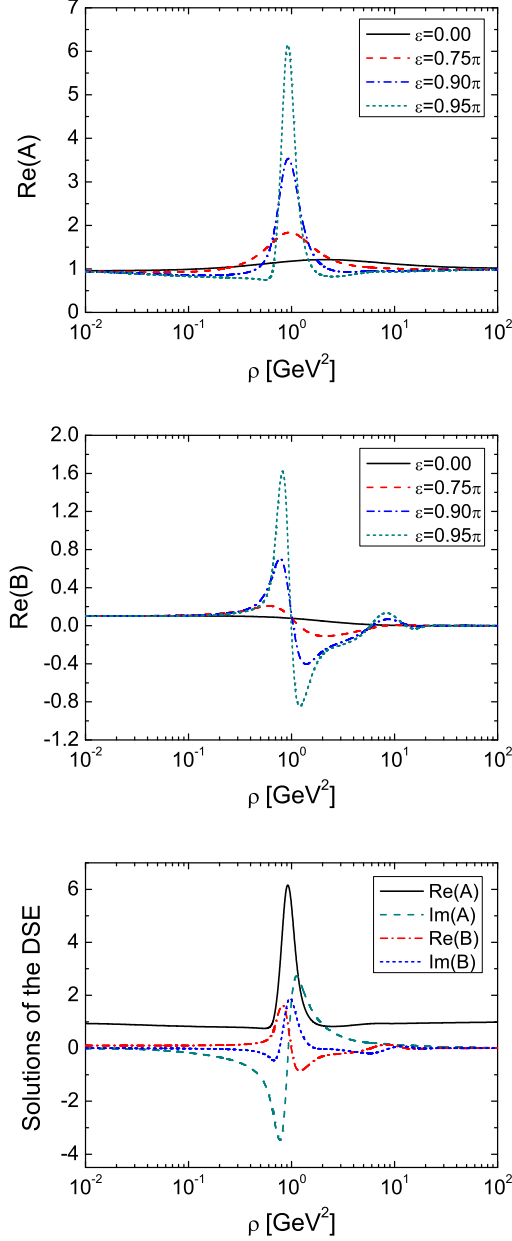


FIG. 6: [color online] Calculated results of the real part of the functions  $A$  and  $B$  in the quark propagator ( $\text{Re}(A)$  in upper panel,  $\text{Re}(B)$  in middle panel) with several values of parameter  $\varepsilon$  and those of the real and the imaginary parts of the functions  $A$  and  $B$  at  $\varepsilon = 0.95\pi$  (lower panel).

$A$  and  $B$  as  $\varepsilon \rightarrow \pi$ , we need the results with  $\varepsilon$  much more close to  $\pi$ . Up to now, we can not say any definite things about the branch-cut structure of the quark propagator yet. However, there are some indications that the power of the divergence of  $A$  and  $B$  is less than one and there would exist branch-cut on the time-like axis. For instance, we find that the absolute values of  $A$  and  $B$  involve oscillation when  $p^2 < p_g^2$  which seems to come from logarithmic divergence. Another evidence is the sym-

metric behavior shown in Fig. 5. Since Fig. 5 is drawn in  $\ln p^2$ , the symmetric feature around the pole also supports logarithmic divergence. However, this issue needs further investigation.

Then, from the behaviors of the functions  $A$  and  $B$ , we can obtain the quark propagator. The obtained results of the  $|\sigma_B|$  in the whole momentum region with the reachable maximal  $\varepsilon$  ( $= 0.95\pi$ ) is shown in Fig. 7. One could easily find from the figure that the Minkowskian quark propagator has one pole at  $p_q^2$ , one zero-point at  $p_g^2$ , respectively. Noting that both the  $A$  and  $B$  are approximately constant at  $p^2 > p_g^2$ , which means that the dynamical mass function  $M(p^2) = B(p^2)/A(p^2)$  also roughly remains constant, one can then recognize that, as the time-like momentum  $p^2 \approx -M^2(0) > p_g^2$ , the structure  $1/[p^2 + M^2(p^2)]$  will generate a simple pole. Ob-

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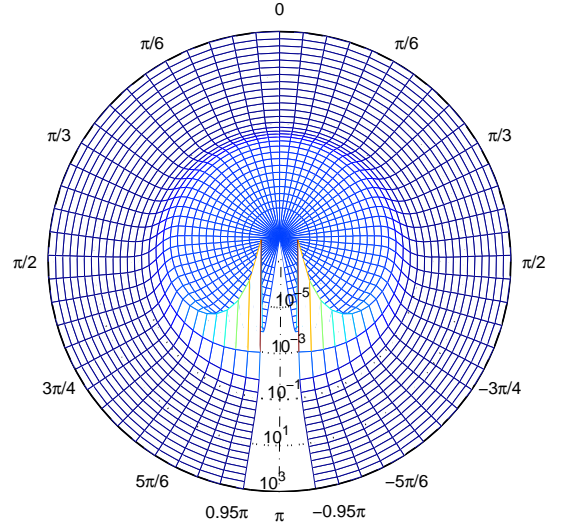


FIG. 7: [color online] Calculated result of the variation behavior of the function  $|\sigma_B|$  in the whole momentum area in the case of  $\varepsilon = 0.95\pi$ .

It should be emphasized that the simple pole of the Minkowskian quark propagator will likely lead to an unconfined quark. Although the result is explicitly against the property of confinement, we tend to think that it is caused by the specific model for the gluon propagator. Actually, the gluon model adopted in this paper is not a rigorously confined one, because the linear confinement potential requires  $\Delta \rightarrow 0$ . Then, we wonder what will happen when we decrease the parameter  $\Delta$  and maintain the mass gap  $M(0)$ . The obtained results at several values of the  $\Delta$  are presented in Table I. It is evident that  $p_g^2$  approaches  $p_q^2$  with the decreasing of  $\Delta$ . It is a positive signal for our assumption that  $\Delta \rightarrow 0$  will lead to  $p_g^2 \rightarrow p_q^2$ . However, we could not prove this statement



yet, since numerical difficulty prevents us from decreasing the  $\Delta$  to infinitesimal. In this article, we leave this question open. However, given such an assumption, we could expect that, as  $p_g^2 \rightarrow p_q^2$ , the simple pole and the zero point of the quark propagator will combine into a lower-order pole (fractional power or logarithmic divergence) which finally confines quark.

TABLE I: Calculated results of the parameter dependence of the Minkowskian quark propagator

$D$ (GeV <sup>2</sup> )	$\Delta$ (GeV <sup>2</sup> )	$p_g^2$ (GeV <sup>2</sup> )	$M(0)$ (GeV)	$p_q^2$ (GeV <sup>2</sup> )
1.06	0.50	-0.88	0.11	-0.012
0.66	0.30	-0.58	0.11	-0.012
0.25	0.10	-0.22	0.11	-0.012

## VI. SUMMARY AND REMARKS

When we have interests with the quantities, such as conductance and viscosity, which involve transport processes, the Minkowskian metric would be inevitable. The traditional method to realize the Minkowskian metric is the so-called analytic continuation of Green function. In this paper, we argue that the meaningful analytic continuation of Green functions requires an identical vacuum. It should be true for any rigorous quantum field theory, but not always promised by models. Besides this problem, the widely used method for analytic continuation in various models is based on the traditional *Källén-Lehmann* spectral representation. However, in some circumstances, especially for quark fields, the situation would become more subtle and complicated. In vacuum, since quark is confined, quark propagator may not have standard *Källén-Lehmann* spectral representation. More or less, such a difficulty may even exist in hot/dense quark matter where the quark degree of freedom becomes important. Therefore, it will invalidate the method of analytic continuation based on the *Källén-Lehmann* spectral representation. It has also been emphasized that, if we implement the translation  $p_0 \rightarrow p_0 - i\mu$  to perform the analytic continuation, the range of  $(p_0 - i\mu)^2 + \vec{p}^2$  is bounded by a parabolic curve. Even if singularities only appear at time-like real axis, such parabolic curve will inevitably encounter the singularities as  $\mu$  become moderately large. In this case, the available analytic information is only that in the inner part of the parabola whose top approaches the first singularity.

In this paper, we have developed a new scheme which is free of all the problems mentioned above. A well-behavior effective gluon propagator and the Ball-Chiu vertex ansatz can promise that the singularities can only appear at time-like axis, thus promise an uniform vacuum

under Euclidean and Minkowskian metrics. Furthermore, by establishing a parameterized metric  $g^{\mu\mu}(\varepsilon)$  which connects the Euclidean metric with the Minkowskian one, we can directly solve the Green function in the full  $p^2$  plane except the time-like axis without implementing any priori decomposition rules such as the *Källén-Lehmann* spectral representation. After confirming the correctness of our parameterized metric scheme in the simple  $\phi^4$  model and in the calculation of electron self-energy at one-loop level and checking the validity of the algorithms for solving the quark DSE in the case of a relative small parameter  $\varepsilon$  ( $= 0.75\pi$ ), we continuously increase the  $\varepsilon$  and obtain the almost Minkowskian quark propagator with  $\varepsilon = 0.95\pi$ . We find that the Minkowskian quark propagator has one simple pole and one zero-point at  $p_q^2$  and  $p_g^2$  respectively. By observation on the dependence of the quark propagator on the screening width  $\Delta$  in the effective gluon propagator, we could expect that, as  $\Delta \rightarrow 0$ , the simple pole and the zero point of quark propagator will combine into a lower-order pole (fractional power or logarithmic divergence) which finally confines quark. However, the exact structure (for instance, branch-cut) has not yet been identified due to the limited  $\varepsilon$ . In hot/dense strong interaction matter where the quark degree of freedom becomes more significant and practical [2, 3, 7, 8, 18, 19], our method need further modifications which is under way. In this case, an interesting question is whether the so-called deconfined quark has the standard *Källén-Lehmann* spectral representation. Anyway, an opposite answer can only come from the non-perturbative nature.

In addition, if one makes use of a gluon propagator model, whose ultraviolet behavior being consistent with the perturbative result, the loop integral in quark DSE will be divergent. In this case, proper regularization would be inevitable. To be honest, we have no idea about how to take dimensional regularization in the numerical calculation at present stage. Since  $p^2$  is a complex number in the calculation, the regularization set by  $p^2 < \Lambda^2$  is also invalid. One might take  $|p^2| < \Lambda^2$  alternatively. However, it is a non-analytic cut-off and inconsistent with our method. In our view, Pauli-Villars regularization procedure could be a reasonable choice, as we have taken in section II, which can overcome the difficulties mentioned above. The related investigation is in progress.

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- [1] E. Peskin and V. Schoroeder, *An Introduction to Quantum Field Theory* (Westview Press, USA, 1995).
  - [2] M. Gyulassy, and L. McLerran, Nucl. Phys. A **750**, 30 (2005); E. Shuryak, Nucl. Phys. A **750**, 64 (2005). E. Shuryak, Prog. Part. Nucl. Phys. **62**, 48 (2009).
  - [3] A. Schafer, Xin-Nian Wang, and Ben-Wei Zhang, Nucl. Phys. A **793** 128 (2007); E. Shuryak, Prog. Part. Nucl. Phys. **62**, 48 (2009).
  - [4] M. S. Bhagwat, P. C. Tandy, AIP Conf. Proc. **842**, 225 (2006).
  - [5] C. D. Roberts, A. G. Williams, Prog. Part. Nucl. Phys. **33**, 477 (1994). R. Alkofer, and L. von Smekal, Phys. Rep. **353**, 281 (2001); C. D. Roberts, M. S. Bhagwat, A. Hoell, S. V. Wright, Eur. Phys. J.- ST **140**, 53 (2007); C. D. Roberts, Prog. Part. Nucl. Phys. **61**, 50 (2008).
  - [6] R. Alkofer, W. Detmold, C. S. Fischer, P. Maris, Phys. Rev. D **70**, 014014 (2004).
  - [7] Huan Chen, Wei Yuan, Lei Chang, Yu-Xin Liu, Thomas Klahn and C. D. Roberts, Phys. Rev. D **78**, 116015 (2008).
  - [8] C. D. Roberts, S. M. Schmidt, Prog. Part. Nucl. Phys. **45**, S1 (2000). P. Maris, and C. D. Roberts, Int. J. Mod. Phys. **12**, 297 (2003);
  - [9] H.S. Zong, L. Chang, F.Y. Hou, W.M. Sun, and Y.X. Liu, Phys. Rev. C **71**, 015205 (2005). F.Y. Hou, L. Chang, W.M. Sun, H.S. Zong, and Y.X. Liu, Phys. Rev. C **72**, 034901 (2005).
  - [10] N. I. Ioakimidis, K. E. Papadakis, E. A. Perdios, BIT **31**, 276 (1991).
  - [11] C. S. Fischer, R. Alkofer, Phys. Rev. D **67**, 094020 (2003).
  - [12] D. C. Curtis, M. R. Pennington, Phys. Rev. D **42**, 4165 (1990).
  - [13] T. Kugo, arXiv: *hep-th/95511033*;
  - [14] D. Zwanziger, Nucl. Phys. B **364**, 127 (1991).
  - [15] R. Alkofer, C. S. Fischer, L. V. Smekal, Prog. Part. Nucl. Phys. **50**, 317 (2003).
  - [16] J. S. Ball, T.-W. Chiu, Phys. Rev. D **22**, 2550 (1980).
  - [17] L. Chang, and C. D. Roberts, Phys. Rev. Lett. **103**, 081601 (2009).
  - [18] A. Bender, D. Blaschke, Y. Kalinovsky, C. D. Roberts, Phys. Rev. Lett. **77**, 3724 (1996).
  - [19] D. Blaschke, C. D. Roberts, Nucl. Phys. A **642**, 197 (1998).